Exam sets October 2023, with solutions

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Please do not forget to write down your name and student number. Good luck!

0. Write down (no proof required) all elements of the set

$$(\{2,4,6\} \cup \emptyset) \cap \{6, \{\emptyset\}, 8\}$$

   The only element in the intersection is 6.

1. Prove that for all sets A, B, C we have $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.
   Suppose $x \in A \setminus (B \cap C)$ then $x \in A$ and x is not in $B \cap C$ so x is not an element of both B and C. To show that $x \in (A \setminus B) \cup (A \setminus C)$ we need to show that either $x \in A$ and not in B or $x \in A$ and not in C or both. It is clear from our assumptions that $x \in A$ so we just need to prove that x is either not in B or not in C. Suppose for a contradiction that it is then x would be both in B and in C so $x \in B \cap C$ but this is impossible because we assumed $x \notin B \cap C$. Our conclusion is that $x \in A \setminus B$ or $x \in A \setminus C$ so $x \in (A \setminus B) \cup (A \setminus C)$.

2. Prove by induction that if $f : X \to X$ is invertible then for every $n \in \mathbb{N}$ the n-times repeated composition $f^{(n)}$ is also invertible.
   Define the statements S_n to say that $f^{(n)}$ is invertible. By definition $f^{(0)}$ is the identity on X which is certainly invertible (it is its own inverse) so S_0 is true. Next assume S_n is true for some $n \in \mathbb{N}$. Then $f^{(n)}$ is invertible and since f is also invertible the composition $f^{(n+1)} = f \circ f^{(n)}$ is also invertible we showned that $f^{(n+1)}$ is invertible so S_{n+1} holds. This completes the induction proof showing that S_n holds for all $n \in \mathbb{N}$.

3. Give an explicit example of a function $f : \{0,1,2\} \to \{0,1,2\}$ that is not injective and not surjective. You are required to prove that your chosen f is not injective and not surjective.
   The constant function $C_0 : [3] \to [3]$ defined by $C_0(x) = 0$ is neither injective nor surjective. It is not injective because $C_0(0) = C_0(1) = 0$ and not surjective because there is no $x \in [3]$ such that $C_0(x) = 1$ because by definition $C_0(x) = 0 \neq 1$.

4. Consider a set X and a subset $A \subseteq X$ and define the function $F : 2^X \to 2^X$ by $F(W) = W \cap A$. Prove that the range of F is 2^A.
   The range of F is

$$F(2^X) = \{S \in 2^X : \text{ There is a } W \in 2^X \text{ such that } S = F(W)\}$$

   First we will show that $F(2^X) \subseteq 2^A$. And then we will show that $2^A \subseteq F(2^X)$. Together this will show that $F(2^X) = 2^A$ by the double inclusion lemma. To show $F(2^X) \subseteq 2^A$ take $S \in F(2^X)$ so there is some W with $S = F(W) = A \cap W$ so $S \subseteq A$, meaning $S \in 2^A$. Next to prove $2^A \subseteq F(2^X)$ pick any $U \in 2^A$ so $U \subseteq A$ and so also $U \subseteq X$. It follows that $F(U) = U \cap A = U$ because $U \subset A$ so $U \in F(2^X)$ concluding the proof.

5. **Point out why the following is incorrect:** Suppose Y is a set with precisely four elements and R is an equivalence relation R on Y such that each equivalence class contains precisely two elements. Since the set of equivalence classes $Y/R = \{\bar{y} : y \in Y\}$ form a partition of Y we have

$$4 = \#Y = \#(\bigcup_{y \in Y} \bar{y}) = \sum_{y \in Y} \#\bar{y} = \sum_{y \in Y} 2 = 8$$

In the middle equality sign we over-count the equivalence classes: each equivalence class is counted precisely twice. For example if $\bar{y} = \{y, x\}$ then $\bar{y} = \bar{x}$ is the same equivalence class but if $x \neq y$ it gets counted twice in the sum.

6. If $f : \{0, 1, 2\} \to \{a, b, c\}$ is defined by $f(0) = a$, $f(1) = b$, $f(2) = a$, write down explicitly all the elements of the inverse image $f^{-1}(\{a, c\})$. Explain your answer.
By definition $f^{-1}(\{a, c\}) = \{x \in [3] : f(x) \in \{a, c\}\}$ since $f(0) = f(2) = a$ we have $f^{-1}(a, c) \supseteq \{0, 2\}$ since $f(1) = b \notin \{a, c\}$ we find $f^{-1}(a, c) = \{0, 2\}$.

7. In the text we proved that if A and B are countably infinite sets then $A \times B$ is also countably infinite. Use this fact to prove that if A, B, C, D are countably infinite sets then $(A \times B) \times (C \times D)$ is countably infinite.
Since A, B are countably infinite (abbreviated c.i.) we have $F = A \times B$ is c.i. also B, C are c.i. so $G = B \times C$ is c.i. Using the same result once more we have $F \times G$ is also c.i. as we were required to prove.

8. Recall $[6] = \{0, 1 \ldots, 5\}$. Define

$$R = \{(x, y) \in [6]^2 : x + y \geq 5\} \cup \{(x, y) \in [6]^2 : x = y\}$$

Is R an equivalence relation on $[6]$? Explain.
R is not an equivalence relation because $(0, 5)$ and $(4, 1)$ are in R but $(0, 1)$ is not in R contrary to the transitivity property of equivalence relations.

9. Prove that if K, L, M are sets and K and L have the same cardinality then M^K has the same cardinality as M^L.
By definition there is an invertible function $h : K \to L$. We aim to show that there is also an invertible function $F : M^K \to M^L$ because that would show that they have the same cardinality. Given $f \in M^K$ define $F(f) : L \to M$ by $F(f) = f \circ h^{-1}$. To show F is invertible we claim $G : M^L \to M^K$ defined by $G(u) = u \circ h$ is its inverse. Indeed: $(G \circ F)(f) = G(f \circ h^{-1}) = f \circ h^{-1} \circ h = f$ so $G \circ F = \mathrm{id}_{M^L}$. Also likewise $(F \circ G)(u) = F(u \circ h) = u \circ h \circ h^{-1} = u$ so $F \circ G = \mathrm{id}_{M^K}$ completing the proof.